

Causal perturbation theory in general FRW cosmologies II: superhorizon behaviour and initial conditions

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Abstract

We describe how causality and energy-momentum conservation constrain the superhorizon behaviour of perturbation variables in a general FRW spacetime. The effect of intrinsic curvature upon the horizon scale is discussed and ‘white noise’ results for the power spectra of generic ‘causally-seeded’ superhorizon perturbations are obtained. We present a detailed derivation of the superhorizon energy density power spectrum for curved universes, and provide elegant mathematical arguments for a curved universe analogue of the Bessel function addition theorem. This yields physical insight into both the familiar k^4 result for flat space and the $K \neq 0$ case: curvature effects are seen to be evident from as early as half the curvature scale. We consider the implications of these results for the physically and numerically significant issue of setting consistent initial conditions.

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I. INTRODUCTION

The standard flat universe inflationary plus dark energy paradigm appears to be remarkable accord with recent CMB experiments [1]. Nevertheless, the confrontation with observation remains indecisive, not only because of the significant experimental uncertainties, but also because good quantitative accuracy has not yet been achieved for competing paradigms such as actively sourced theories, which have so far only been extensively studied for cosmic defects in flat FRW ($K = 0$) backgrounds [2–5]. Moreover, even for flat universes, a subsidiary role for defect networks complementing the inflationary power spectrum cannot be excluded [6]. The possibility of searching for distinct defect signatures in the CMB, such as nonGaussianity [7] or in the polarisation spectra [8] may go some way toward resolving these remaining uncertainties. However, in order to have confidence in cosmological parameter estimation with a purely inflationary paradigm, it will be necessary to constrain the alternative models, including both ‘primordial’ and ‘active’ perturbation sources, as well as the effects of vector and tensor modes and $K \neq 0$ backgrounds. Here the combination of intrinsic curvature and defect sources is particularly interesting, motivating a study of the superhorizon behaviour and the choice of consistent initial conditions for such perturbations. Of particular interest is the possibility of a testable signature of superhorizon fall-off in ‘causal’ or ‘active’ models in the CMB polarisation power spectrum.

The Cauchy problem for cosmological perturbations over a $K = 0$ FRW background has been extensively discussed in the literature. Using energy-momentum conservation, and assuming that the early universe was purely homogeneous and isotropic, one can show [9–12] that the power spectrum of the stresses should be that of white noise ($\propto k^0$), while that of the density should be suppressed as k^4 , for wavelengths much larger than the horizon; here, k is the comoving wavenumber so this the superhorizon condition is $k \ll aH$ where $H = \dot{a}/a$ and a is the scale factor. Assuming that matter undergoes random displacements of length d_c at each point in space, while satisfying energy-momentum conservation, one can estimate that the density power spectrum turns over to k^4 behaviour at $k_c \sim 2d_c^{-1}$ [9]. This result may be used [12,3] to simplify the calculations to determine consistent initial conditions for numerical analyses. One assumes that all perturbations of relevance today come from far outside the initial horizon ($k\tau_i \ll 1$) so that one may argue that the strong k^4 suppression allows us to set all these variables to zero initially.

For the general $K \neq 0$ FRW cosmologies, these issues remain unaddressed, although there do exist several analyses of the effect of curvature on the CMB assuming primordial inflationary perturbations described by power law ansatzes. Some examples include: the generalisation of the Sachs-Wolfe effect to include curvature effects [11]; refs. [13] and [14] treat the open universe case, respectively using a gauge invariant and a synchronous gauge formalism — see also ref. [15] — and ref. [16] investigated the closed universe case, again using a gauge invariant approach. However, the literature does not contain (to our knowledge) any direct analysis of the superhorizon effects of energy-momentum conservation on active and causal seed sources in intrinsically curved FRW spacetimes.

In this paper we shall therefore describe the manner in which causality and energy-momentum conservation constrain the superhorizon behaviour of the perturbation vari-

ables in the general FRW spacetime, in curvilinear coordinates. As discussed in our previous paper — ref. [17] — we may describe the symmetry properties and stress energy conservation laws of these spacetimes in two equivalent languages corresponding to integral and differential conservation laws. The language of conserved vector densities \hat{I}^μ (*c.f.* Katz *et al* [18]) uses the conformal geometry of the background spacetime to naturally construct integral constraint equations, representing strong conservation laws defined over some region of a spacelike hypersurface. These reduce to Noether conservation laws when constructed using the Killing geometry of the background.

Traschen [19] obtained such integral constraints using as a background manifold a maximally symmetric de Sitter spacetime in quasi-Minkowskian coordinates, and these were related to the vector densities \hat{I}_ξ^μ by Dereulle *et al* [20]. However, in order to make use of these integral constraints, we need to know their behaviour on the boundary of the integration volume. In the case of bounded (or “local”) perturbations, the boundary terms are assumed to vanish, that is, these previous analyses consider perturbations whose stress energy vanishes beyond the horizon. This is a stronger assumption than required by the physics, since only the two-point correlators between these quantities have to vanish in order to satisfy causality. In other words, a physical assumption has been made that is perhaps better suited to the analysis of an explosive event rather than more general causal perturbations, such as those seeded by topological defects, which may persist beyond the horizon (*e.g.* cosmic strings). Moreover, the choice of background relative to which energy-momentum is formally defined is that of a de Sitter universe, rather than the FRW background convenient for most simulations.

Hence, we turn to the equivalent language of energy-momentum pseudo-tensors [17], outlined in §II. Again, the relative simplicity of the flat space case is complicated by the introduction of curvature. In §III A we define the two-point correlator, and discuss causality, while the transformed correlator (with respect to the eigenfunctions of the Laplacian) is described in §III B. The effect of intrinsic curvature upon the horizon scale is dealt with in §IV A; and in §IV B we obtain results for a general FRW spacetime analogous to the white noise power spectrum of two arbitrary functions in flat space. Then, in §V, we consider in more detail the superhorizon behaviour of the pseudo-energy. This not only yields the familiar k^4 behaviour for the isotropic (00-)mode on superhorizon scales for $K = 0$, but is also well suited to study the $K \neq 0$ case. We regain the flat space results in the local limit and discuss the effects of intrinsic curvature on the isotropic mode in a Helmholtz decomposition of the density perturbation associated with causal sources, for both open and closed cosmologies. In §VI we comment on the implications of these results for the choice of consistent initial conditions. We work in the synchronous gauge because of its ubiquity in numerical simulations and the greater physical transparency offered by this gauge choice.

II. ENERGY-MOMENTUM CONSERVATION

We shall consider metric perturbations $h_{\mu\nu}$ about a general FRW spacetime

$$ds^2 = a^2(\gamma_{\mu\nu} + h_{\mu\nu})dx^\mu dx^\nu, \quad (1)$$

where the comoving background line element in ‘conformal-polar’ coordinates $(\tau, \chi, \phi, \theta)$ is given by

$$\gamma_{\mu\nu} dx^\mu dx^\nu = -d\tau^2 + \frac{1}{|K|} \left[d\chi^2 + \sin_K^2 \chi \left(d\theta^2 + \sin^2 \theta d\phi^2 \right) \right] , \quad (2)$$

with the function $\sin_K \chi$ depending on the spatial curvature K as

$$\sin_K \chi = \begin{cases} \sinh \chi , & K < 0 , \\ \chi , & K = 0 , \\ \sin \chi , & K > 0 . \end{cases} \quad (3)$$

We shall adopt the synchronous gauge defined by the choice $h^{0\mu} = 0$, so that the trace is given by $h \equiv h^i_i$ (with the convention throughout that Greek indices run from 0 to 3 and Latin from 1 to 3).

Here, $a \equiv a(\tau)$ is the scalefactor, for which we can define the conformal Hubble factor $\mathcal{H} = \dot{a}/a$, with dots denoting derivatives with respect to conformal time τ . The evolution of these background quantities is described by the Friedmann equations

$$\kappa a^2 \rho = 3(H^2 + K), \quad \kappa a^2 (\rho + 3p) = -6\dot{H} , \quad (4)$$

where the background tensor $\bar{T}_{\mu\nu} = (\rho, p, p, p)$ includes the dark energy of the universe (or cosmological constant). In an earlier paper, [17], we have shown that one may use the Bianchi identities to rewrite the Einstein equations $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa T_{\mu\nu}$ (with $\kappa = 8\pi G$), to obtain a description of the perturbed energy-momentum in terms of a linear tensor

$$\begin{aligned} \kappa \tau^0_0 &= a^2 \delta G^0_0 + \mathcal{H} \dot{h} - K h , & \kappa \tau^0_i &= \delta G^0_i , \\ \kappa \tau^i_j &= \delta G^i_j - \mathcal{H} [\dot{h}^i_j - \dot{h} \delta^i_j] . \end{aligned} \quad (5)$$

Note that here we have dropped the deltas in the original definition, understanding that the τ^μ_ν are the perturbed parts of the energy-momentum pseudo-tensor only. We have also separated the perturbed energy-momentum tensor into two parts: the first order part $\delta T_{\mu\nu}$ incorporates the stress energy of the radiation fluid, baryonic matter, and cold dark matter, and a contribution $\Theta_{\mu\nu}$ representing the stress tensor of an evolving defect network or some other causal sources. This is assumed to be small (of order $\delta T_{\mu\nu}$) and ‘stiff’, that is, its energy and momenta are conserved independently of the rest of the matter and radiation in the universe and to lowest order its evolution is unaffected by the metric perturbations $h_{\mu\nu}$.

The components (5) may be simply related to conserved vector densities \hat{I}^μ obtained by defining energy and momenta with respect to a general FRW background [17], and therefore represent a locally valid definition of these quantities. The conservation equations for these quantities are

$$\tau^0_{0,0} + \tau^i_{0|i} = 0 , \quad \tau^0_{i,0} + \tau^j_{i|j} = 0 , \quad (6)$$

where the bar denotes covariant differentiation with respect to the spatial three metric γ_{ij} .

The pseudo-energy τ_0^0 is a particularly useful variable for tracking the growth of large-structure which is why we focus on it in this paper. In ref. [17], we showed that pseudo-energy is proportional to the growing mode of the density perturbation on superhorizon scales whether the universe is dominated by radiation, matter or curvature (see also [12,3]). In addition, we briefly discussed the implications of matching conditions at an instantaneous phase transition, with particular emphasis on the choice of consistent initial conditions. However, a more thorough treatment requires that we consider not the perturbations themselves, but rather their correlators.

III. TWO-POINT CORRELATORS

A. Two-point correlators and causality

Causality is usually defined in terms of two-point correlation functions, which contain all the information pertinent to Gaussian perturbations [22]. If two random functions $b_{\mu\nu}$ and $d_{\mu\nu}$ are assumed to be statistically spatially homogeneous and isotropic, then their two point correlation function, $C_{\mu\nu\sigma\lambda}(\mathbf{y}, \tau, \tau') = \langle b_{\mu\nu}(\mathbf{x}, \tau) d_{\sigma\lambda}(\mathbf{x} + \mathbf{y}, \tau') \rangle$ is a function of time and the ‘radial’ coordinate $|\mathbf{y}| = \chi_y$ only. The expansion of the universe breaks time translation invariance [5], so that we may expect dependence on both τ and τ' , rather than just on their difference $\tau - \tau'$. As we are constructing a causal theory of cosmological perturbations, we note that causality imposes the constraint

$$C_{\mu\nu\sigma\lambda}(\mathbf{y}, \tau, \tau') = \langle b_{\mu\nu}(\mathbf{x}, \tau) d_{\sigma\lambda}(\mathbf{x} + \mathbf{y}, \tau') \rangle = 0, \quad \text{for } \chi_{\mathbf{y}} > \tau + \tau', \quad (7)$$

so that the two-point correlation function has compact support in \mathbf{y} . Here we may assume without loss of generality that $\tau > \tau'$. The fact that the correlator has compact support in turn means that the Fourier transform (in flat space) of the correlator is analytic, as are all the integral transforms with which we shall be concerning ourselves.

In particular, we shall be concerned with the power spectra of various functions, defined as the (Fourier or eigenfunction) coefficient of the equal time two point correlator of the function with its complex conjugate:

$$C_{\mu\nu\mu\nu}(\mathbf{y}, \tau, \tau) = \langle b_{\mu\nu}(\mathbf{x}, \tau) b_{\mu\nu}^*(\mathbf{x} + \mathbf{y}, \tau) \rangle = \frac{1}{V} \int dV_{\mathbf{x}} b_{\mu\nu}(\mathbf{x}, \tau) b_{\mu\nu}^*(\mathbf{x} + \mathbf{y}, \tau), \quad (8)$$

for an arbitrary function $b_{\mu\nu}$. Implicit in the final equality of (8) is the “ergodic hypothesis” which states that one may interchange spatial and ensemble averages. In this case the assumption of causality may be expressed as

$$C_{\mu\nu\mu\nu}(\mathbf{y}, \tau, \tau) = 0, \quad \text{for } |\mathbf{y}| > 2\tau = 2\chi_H. \quad (9)$$

In the remainder of this paper we shall frequently suppress the (equal) time dependence of the correlators.

B. The Helmholtz decomposed correlator

For perturbations over a curved FRW background, it is usual to employ the Helmholtz decomposition using the linearly independent eigenfunctions of the Laplacian in polar coordinates [15,23]. We expand all perturbation quantities in terms of the eigenfunctions $\mathbf{Q}^{(m)}$, which are the scalar ($m = 0$), vector ($m = \pm 1$) and tensor ($m = \pm 2$) solutions to the Helmholtz equation

$$\nabla^2 \mathbf{Q}^{(m)} \equiv \gamma^{ij} \mathbf{Q}_{|ij}^{(m)} = -k^2 \mathbf{Q}^{(m)}, \quad (10)$$

where the eigentensor has $|m|$ suppressed indices (equal to the rank of the perturbation). The explicit representations of these eigenfunctions are given in Appendix A. They are conveniently labeled in terms of a generalised wavenumber q , and its normalised equivalent $\beta = q/\sqrt{|K|}$, related to k via

$$q^2 = k^2 + (|m| + 1)K, \quad \Longleftrightarrow \quad \beta^2 = \tilde{k}^2 + (|m| + 1)\frac{K}{|K|}, \quad (11)$$

where $k = \sqrt{|K|}\tilde{k}$ and, in flat space, is the Fourier wavenumber.

The spectra for flat and open universes ($K \leq 0$) are continuous and complete for $\beta \geq 0$. For the open universe, this means that the normalised scalar eigenvalues are greater than or equal to unity: $\tilde{k}^2 \geq 1$. For the closed ($K > 0$) case, the spectrum is discrete because of the existence of periodic boundary conditions. For scalar perturbations, we then have $\beta = 3, 4, 5, \dots$ since the $\beta = 1, 2$ modes are pure gauge [24]. These spectra yield a set of eigenfunctions that is complete, orthonormal, and sufficient to expand all perturbations produced in a finite region.

The literature (see, for instance, refs. [23,25,26]) contains several comments about the open universe bound $\tilde{k}^2 \geq 1$, which could be misinterpreted as implying that perturbations can only be generated below the curvature scale. However, the particle horizon can exceed the curvature scale (see later) which means that, in principle, causal perturbations also can be generated on these large lengthscales. We shall now briefly place these comments in the context of more general eigenfunction decompositions. If one is attempting to describe the stochastic properties of inflationary cosmologies it may be desirable [25,26] to include the eigenfunctions associated with $0 \leq \tilde{k}^2 < 1$ and even $\tilde{k}^2 < 0$, as these may produce effects in the CMB anisotropy patterns — the Grischuk-Zel'dovich effect [27]. However, this is only necessary if the assumption that the inflaton field is in the vacuum breaks down for some correlation length much larger than the curvature scale and the effect apparently is not measurable in multi-stage inflation models [28]. Moreover, these eigenfunctions may be obtained only by analytic continuation and are not orthonormal, nor are they linearly independent of the $\tilde{k}^2 \geq 1$ modes. For square integrable functions — such as those describing the correlators of causal perturbations — the eigenfunctions labelled by $\tilde{k}^2 \geq 1$ are sufficient.

The $\tilde{k}^2 \geq 1$ and $\tilde{k}^2 < 1$ modes are often referred to as the ‘subcurvature’ and ‘supercurvature modes’, because their apparent characteristic wavelength ($\propto 1/\tilde{k}$) is (respectively) smaller or larger than the comoving curvature scale. However, this nomenclature is misleading: the $k^2 \geq 1$ modes can contain information about effects above the curvature

scale, in which case they are very different to their flat space counterparts. These sub-curvature modes are *not* therefore confined to subcurvature scales. They are complete in that they can be used to describe a perturbation on any lengthscale provided it has compact support. Now since the horizon can exceed the curvature scale, this completeness of the $\tilde{k} > 1$ modes ensures their sufficiency in describing the correlations of causal perturbations on all lengthscales and at all times.

In an FRW universe, the power spectrum (8) of any scalar function of \mathbf{y} in terms of the scalar eigenfunctions $\zeta_{\beta lm} = \Phi_{\beta}^l(\chi_{\mathbf{x}})Y_{lm}(\theta_{\mathbf{x}}, \phi_{\mathbf{x}})$, where we have decomposed the $Q^{(0)}$ using spherical harmonics (see (A11)). This amounts to a transform on the radial coordinate, and a series expansion on the angular part. In particular we wish to expand the power spectrum of the contracted second derivative of the pseudo-stresses $\tau_{j|i}^{i|j}$ given in (5) as

$$\langle \tau_{j|i}^{i|j}(\mathbf{x}) \tau_{q|p}^{p|q}(\mathbf{x} + \mathbf{y}) \rangle = \sqrt{\frac{2}{\pi}} \sum_{l,m} \int d\mu(\beta) \Phi_{\beta}^l(\chi_{\mathbf{y}}) Y_{lm}(\theta_{\mathbf{y}}, \phi_{\mathbf{y}}) \langle \tau_{j|i}^{i|j} \tau_{q|p}^{p|q} \rangle_{lm}(\beta),$$

where

$$\begin{aligned} \langle \tau_{j|i}^{i|j} \tau_{q|p}^{p|q} \rangle_{lm}(\beta) &= \sqrt{\frac{2}{\pi}} \int dV_{\mathbf{y}} \Phi_{\beta}^l(\chi_{\mathbf{y}}) Y_{lm}^*(\theta_{\mathbf{y}}, \phi_{\mathbf{y}}) \langle \tau_{j|i}^{i|j}(\mathbf{x}) \tau_{q|p}^{p|q}(\mathbf{x} + \mathbf{y}) \rangle \\ &= \sqrt{\frac{2}{\pi}} \int dV_{\mathbf{x}} \tau_{j|i}^{i|j}(\mathbf{x}) \int dV_{\mathbf{y}} \tau_{q|p}^{p|q}(\mathbf{x} + \mathbf{y}) \Phi_{\beta}^l(\chi_{\mathbf{y}}) Y_{lm}^*(\theta_{\mathbf{y}}, \phi_{\mathbf{y}}). \end{aligned} \quad (12)$$

Here, in the last expression, we have interchanged integration with respect to \mathbf{x} and \mathbf{y} . In particular, the isotropic 00-mode is given by

$$\begin{aligned} \langle \tau_{j|i}^{i|j}(\mathbf{x}) \tau_{q|p}^{p|q}(\mathbf{x} + \mathbf{y}) \rangle_{00}(\beta) &= \frac{1}{\sqrt{2\pi^2}} \int dV_{\mathbf{y}} \Phi_{\beta}^0(\chi_{\mathbf{y}}) \langle \tau_{j|i}^{i|j}(\mathbf{x}) \tau_{q|p}^{p|q}(\mathbf{x} + \mathbf{y}) \rangle \\ &= \frac{1}{\sqrt{2\pi^2}} \int dV_{\mathbf{x}} \tau_{j|i}^{i|j}(\mathbf{x}) \int dV_{\mathbf{y}} \tau_{q|p}^{p|q}(\mathbf{x} + \mathbf{y}) \Phi_{\beta}^0(\chi_{\mathbf{y}}), \end{aligned} \quad (13)$$

where we have substituted $Y_{00} = 1/\sqrt{4\pi}$.

IV. THE TRANSFORMED HORIZON SCALE AND WHITE NOISE

A. The transformed horizon scale

In order to develop intuition as well as to investigate the curved universe superhorizon modes in a Helmholtz decomposition, we need to establish the relationship between β and the comoving horizon scale $\chi_H = \tau$. This is analogous to the flat space result $k\tau \approx 3\pi/2$ for a Fourier-Bessel mode with a wavelength of the order of the horizon. We therefore consider the expansion, in terms of eigenfunctions of the Helmholtz equation, of a ‘top hat’ window function that vanishes for $\chi_{\mathbf{x}} > \chi_H$, and is given by $A(\mathbf{x}) = A$

for $\chi_{\mathbf{x}} \leq \chi_H$, where we choose the arbitrary constant $A = 1/\sqrt{8}$. The fact that $A(\mathbf{x})$ is defined in terms of the radial coordinate only means that we may set $l = 0$ in the Helmholtz decomposition to obtain

$$A(\mathbf{x}) = \frac{1}{\sqrt{2\pi^2}} \int d\mu(\beta) \Phi_{\beta}^0(\chi) A_{00}(\beta), \quad A_{00}(\beta) = \frac{1}{\sqrt{2\pi^2}} \int dV A(\mathbf{x}) \Phi_{\beta}^0(\chi). \quad (14)$$

Now, using $dV = \sqrt{-g} d\chi d\Omega$, $g = \det(g_{ij})$, and appealing to (A8), we have

$$A_{00}(\beta) = \begin{cases} \frac{\sinh^2 \chi_H}{|K|^{3/2} \sqrt{\beta^2 + 1}} \Phi_{\beta}^1(\chi_H) = \frac{1}{|K|^{3/2} \beta^2 + 1} \left[\frac{\sin \beta \chi_H \cosh \chi_H}{\beta} - \sinh \chi_H \cos \beta \chi_H \right], & K < 0, \\ \frac{\chi_H^2}{\beta} \Phi_{\beta}^1 & = \frac{1}{\beta^2} \left[\frac{\sin \beta \chi_H}{\beta} - \chi_H \cos \beta \chi_H \right], & K = 0, \\ \frac{\sin^2 \chi_H}{|K|^{3/2} \sqrt{\beta^2 - 1}} \Phi_{\beta}^1(\chi_H) = \frac{1}{|K|^{3/2} \beta^2 - 1} \left[\frac{\sin \beta \chi_H \cos \chi_H}{\beta} - \sin \chi_H \cos \beta \chi_H \right], & K > 0, \end{cases} \quad (15)$$

where we have used the explicit expression (A5) for Φ_{β}^1 given in Appendix A.

In Fig. 1 we give the transformed ‘top hat’ functions for two values of the horizon χ_H , normalised to $A_{00} = 1$ at $\beta = 0$. When the horizon is less than or equal to the comoving curvature scale ($\chi_H \leq 1$), we see that the open universe transform displays very similar behaviour to the flat space Bessel transform. However, on supercurvature scales ($\chi_H > 1$), the open and flat universe transforms exhibit different behaviour. In particular, we shall choose the first zero of the ‘top hat’ transform as a means of describing the relationship between the horizon scale χ_H and the corresponding value of the transformed variable β_H (the horizon mode wavenumber). From these we will calculate the value of

$$\alpha \equiv \beta_H \chi_H,$$

in order to discuss how this relationship is affected by curvature. The meaning of this relation is that, for a given χ_H , those modes with $\beta \chi_H < \alpha$ will have characteristic wavelengths larger than the horizon. Of course, for flat space and Cartesian coordinates, this relation is $\lambda = 2\pi/k < 2\pi/k_H = \chi_H$.

The first zeros of A_{00} in (15) are given by the solutions to

$$0 = \tan \beta \chi_H - \beta \times \begin{cases} \tanh \chi_H, & K < 0, \\ \chi_H, & K = 0, \\ \tan \chi_H, & K > 0, \end{cases} \quad (16)$$

which are shown in Fig. 2. Note from (15) that the zeros of (16) will be precisely the zeros of Φ_{β}^1 , assuming $\chi_H > 0$ (for $K \leq 0$) and assuming $0 < \chi_H < \pi$ for the closed ($K > 0$) case.

The open and flat universes display phenomenologically similar curves. Certainly, as can be seen from Fig. 2, the curves are approximately parallel for χ_H less than (or of the order of) the curvature scale, and that, on these scales, α is therefore the same ($\sim 3\pi/2$) as for flat space: in the local limit, $\beta \chi = qy \rightarrow kr$ when $\chi \ll 1$ ($y \ll \sqrt{|K|}$) or when $|K| \rightarrow 0$. However, for comoving distances much larger than $\chi_H \sim 4$ the effect of the curvature is to effectively shrink α as the curvature induced volume growth becomes significant. The corresponding limit is

$$\alpha \equiv \beta_H \chi_H \rightarrow \pi \quad \text{as} \quad \chi_H \rightarrow \infty \quad (K < 0).$$

It is clear from this simple discussion that the characteristic lengthscale of a mode corresponding to the eigenvalue k is given by $1/\beta$.

FIGURES

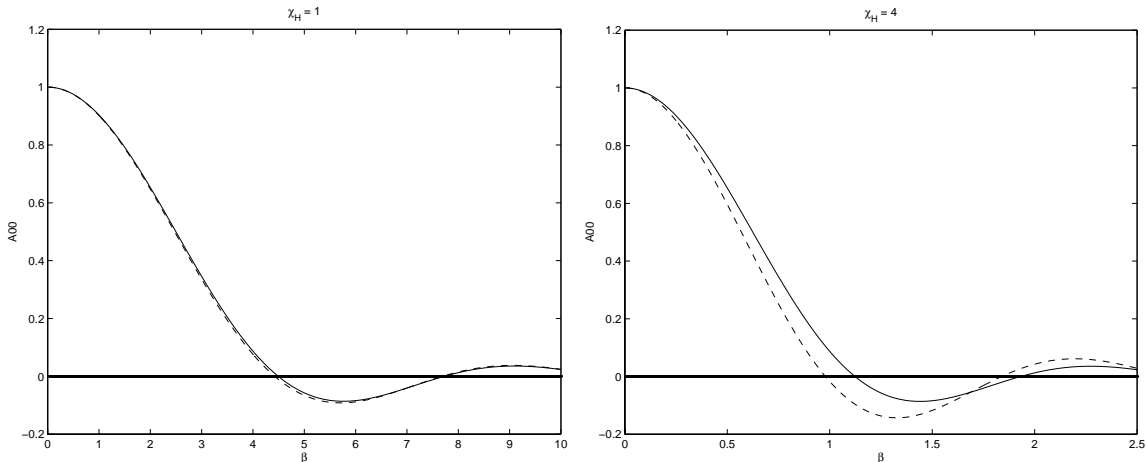


FIG. 1. The transformed ‘top hat’ function A_{00} : $K = 0$ (—), $K = -1$ (---).

For an open universe, on very subhorizon or very superhorizon scales, we may simply take $\beta\tau \gg 1$, or $\beta\tau \ll 1$ respectively, which is directly analogous to the flat space ($\tilde{k}\tau \gg 1$, $\tilde{k}\tau \ll 1$) limits. However, we note that we must not neglect the precise location of the horizon if we are to describe the effect of the curvature scale upon the behaviour of the eigenfunctions. We shall return to this issue later (see §V), when we observe that curvature effects are significant from as early as $\tau = \chi_H = 0.5$.

At this point a brief comment on the difference between the Hubble radius and the particle horizon is relevant. In practice, causal perturbations are usually suppressed beyond the Hubble radius because this is roughly the maximum distance over which correlations can be established in one Hubble time. However, in principle, they can exist out to the particle horizon, so we must consider this distinction carefully (e.g. for cosmic strings or, in a more extreme case, for exploding neutrino shells). From the Friedmann equations (4) we see that the comoving Hubble radius $\mathcal{H}^{-1} = (1 - \Omega)^{1/2} < 1$ for an open universe, where the density parameter $\Omega = \rho/\rho_{\text{crit}}$ and $\rho_{\text{crit}} = 3\mathcal{H}^2/\kappa a^2$. Clearly, this implies that the Hubble radius is always less than the curvature scale. In inflationary scenarios the Hubble radius is usually the most important quantity and this bound has been used to argue that perturbations with characteristic wavelengths smaller than the curvature scale are well modelled by a flat universe (see, for example, ref. [25]). However, the open universe particle horizon is obtained from the equation

$$\sinh^2\left(\frac{\chi_H}{2}\right) = \frac{1}{\Omega} - 1, \quad (17)$$

so that the proper distance $d_H = a\chi_H \approx 2H^{-1}$ for $\Omega \approx 1$, where H^{-1} is the physical Hubble radius. It is therefore possible for the particle horizon to cross the curvature scale: $\chi_H > 1$ for $\Omega < 0.75$. Moreover, in order for $\chi_H > 0.5$ today (when curvature effects become more important — see §V) we require only that $\Omega < 0.94$ today. This is well within current observational constraints, and serves to motivate a more complete treatment of the physics around the curvature scale.

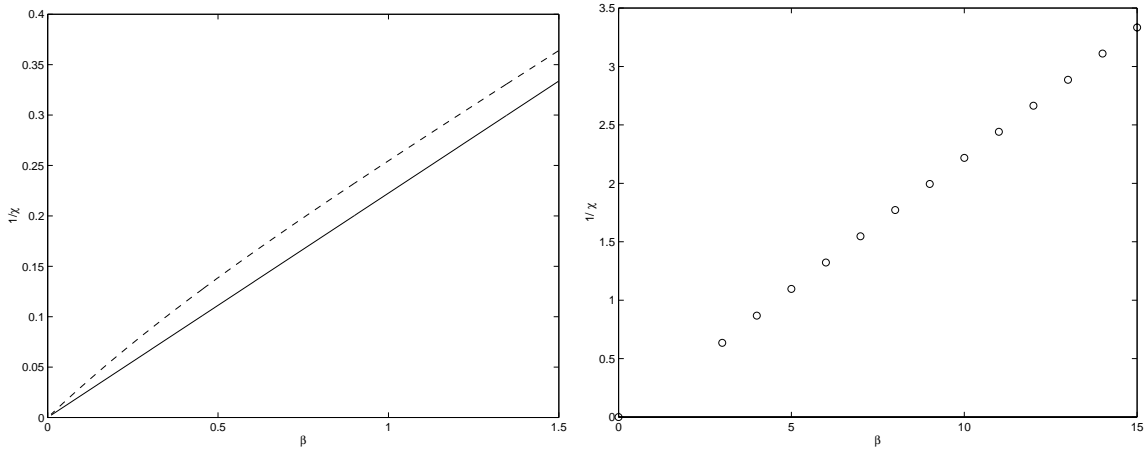


FIG. 2. Solutions to (16): $K = 0$ (—), $K = -1$ (---), $K = +1$ (o) .

For the closed ($K > 0$) case we see that for the discrete spectrum $\beta = 3, 4, \dots$, we have $\alpha \approx 3\pi/2$, which goes over to the (continuous) flat space result in the limit $\chi \rightarrow 0$. We note that the notion of supercurvature and superhorizon perturbations with $\beta \ll \alpha$, $\chi_H \gg 1$ does not have meaning in the closed universe. Indeed, from Fig. 2, we observe that all the β have zeros for values of $\chi_H \leq \pi/2$, at which point the closed universe has its maximal circumference and surface area.

B. The white noise spectrum

We may use causality and the superhorizon limit to establish that the correlators between pseudo-stress must go as a constant on superhorizon scales $\beta\chi_{\mathbf{y}} \ll \alpha$. Substituting the small argument series expansion of $\sin \beta\chi$, in the exact expression (A4) for $\Phi_{\beta}^0(\chi)$, we find that the isotropic ($l = 0$) mode for the correlator of two arbitrary scalar functions may be written as

$$\langle f(\mathbf{x}) \ g(\mathbf{x} + \mathbf{y}) \rangle_{00}(\beta) = \sum_{n=0}^{\infty} \frac{(-1)^n \beta^{2n}}{(2n+1)!} \int dV_{\mathbf{y}} \ < f(\mathbf{x}) \ g(\mathbf{x} + \mathbf{y}) > \times \begin{cases} \frac{\chi_{\mathbf{y}}^{2n+1}}{\sinh(\chi_{\mathbf{y}})} & K < 0 \\ \chi_{\mathbf{y}}^{2n} & K = 0 \\ \frac{\chi_{\mathbf{y}}^{2n+1}}{\sin(\chi_{\mathbf{y}})} & K > 0 \end{cases} . \quad (18)$$

On superhorizon scales, we have $\beta \ll \alpha/\chi_{\mathbf{y}}$ for a given comoving radius $\chi_{\mathbf{y}} \gg \chi_H$ (where the results of §IV A imply that $\pi < \alpha < 3\pi/2$). On sufficiently superhorizon scales, we may therefore choose $\beta \ll 1$, and the lowest order contribution to (18) will be proportional to β^0 . Hence, $\langle f(\mathbf{x}) \ g(\mathbf{x} + \mathbf{y}) \rangle_{00}(\beta) \propto \beta^0$, provided that the integrals are finite, which will be the case if we appeal to causality to set $\langle f(\mathbf{x}) \ g(\mathbf{x} + \mathbf{y}) \rangle = 0$ for large values of the radial variable $\chi_{\mathbf{y}}$. This result confirms those by Avelino [9], Pen *et al* [3] and Veerarghavan and Stebbins [12] for $K = 0$.

Moreover, as the vector and tensor eigenfunctions given in Appendix A are proportional to the scalar radial functions Φ_{β}^l , we can conclude that correlators of vectors and tensors will be premultiplied by a factor behaving as for the scalars. This is in agreement

with the results of Durrer *et al* [5] for $K = 0$, and corrects the common, but erroneous, conclusion that the flat space anisotropic stress power spectra go as k^4 (see, for example, refs. [20,29,33]).

Note that, if we attempt to find the behaviour with respect to k , we run into difficulties in describing the higher order corrections. This is a consequence of the fact that there will be a constant (curvature) contribution to the k^0 term from each term β^{2n} in the series expansion (18) of $\Phi_\beta^0(\chi_{\mathbf{y}})$.

V. SUPERHORIZON BEHAVIOUR OF THE ISOTROPIC MODE

Of the several arguments in the literature which use energy-momentum conservation to yield a k^4 superhorizon power spectrum for flat universes, that provided by Avelino [9] is the most convincing — see also refs. [3,11,12]. Using Cartesian coordinates, and assuming that the volume is large enough to justify the ergodic hypothesis, we may equate spatial and ensemble averages, and Fourier transform the correlator of the stresses, arriving at

$$|\tau_{ij}(k, \eta) \tau_{kl}^*(k, \eta')| = V^{-1} \int_{|\mathbf{y}| < d_c} d^3y e^{i\mathbf{k} \cdot \mathbf{y}} \langle \tau_{ij}(\mathbf{x}, \eta) \tau_{kl}(\mathbf{x} + \mathbf{y}, \eta') \rangle. \quad (19)$$

Here we have implicitly made use of the convolution theorem, and hence the property $e^{i\mathbf{k} \cdot \mathbf{y}} = e^{-i\mathbf{k} \cdot \mathbf{x}} e^{i\mathbf{k} \cdot (\mathbf{x} + \mathbf{y})}$ of the exponential kernel used in Fourier theory, while causality implies that the correlator has compact support in \mathbf{y} . Hence, on superhorizon scales ($|\mathbf{k}|d_c \ll 1$) we may neglect the exponential part of the integrand and find that the “power spectrum” of the pseudo-stresses $\mathcal{P}_{\tau_{ij}} = V|\tau_{ij}|^2/(2\pi)^3$ approaches a constant: we have a white noise (k^0) spectrum. Noting that partial differentiation with respect to x^i is equivalent to multiplying the associated Fourier variable by k_i , we may use energy-momentum conservation (20) to obtain the result:

$$\begin{aligned} \mathcal{P}_{\tau_{00}} = V \frac{|\tau_{00}(k, \eta)|^2}{(2\pi)^3} &= \frac{k_i k_j k_k k_l}{(2\pi)^3} \int_{\eta_i}^{\eta} d\eta_1 \int_{\eta_i}^{\eta_1} d\eta_2 \int_{\eta_i}^{\eta} d\eta'_1 \int_{\eta_i}^{\eta'_1} d\eta'_2 \times \\ &\int_{|\mathbf{y}| < d_c} d^3y \langle \tau_{ij}(\mathbf{x}, \eta_2) \tau_{kl}(\mathbf{x} + \mathbf{y}, \eta'_2) \rangle. \end{aligned} \quad (20)$$

We conclude that, since $\langle \tau_{ij} \tau_{kl} \rangle$ exhibits white noise, then the power spectrum of τ_{00} should approach a k^4 spectrum on superhorizon scales, as has been confirmed in numerical studies [34]. Similarly, the power spectrum of τ_i^0 is proportional to k^2 , in this limit.

The difficulty in generalising the above analysis to the general FRW universe lies in the lack of a useful equivalent to the exponential partial wave function in the general curved ($K \neq 0$) FRW universe (see ref. [27]), as well as in this argument’s dependence upon the choice of a Cartesian coordinate system. However, we can recast this argument using the eigenmodes of the Laplacian to investigate the constraints imposed by energy-momentum conservation and causality in both flat and intrinsically curved FRW universes, regaining also the familiar k^4 spectrum in the local (or $|K| \rightarrow 0$) limit.

A. Eigenmode splitting in curved spherical coordinates

In order to generalise this argument we must be able to split our eigenmodes in spherical coordinates under spatial translations to obtain an analogue of the convolution theorem. For the flat ($K = 0$) FRW universe and spherical coordinates we have the relationship

$$j_0(kr_{\mathbf{y}}) = \sum_{n=0}^{\infty} (2n+1) j_n(kr_{\mathbf{x}}) j_n(kr_{\mathbf{x}+\mathbf{y}}) P_n(\cos \sigma) , \quad (21)$$

where σ is the angle between $-\mathbf{x}$ and $\mathbf{x}+\mathbf{y}$ and we have used $r_{-\mathbf{x}} = r_{\mathbf{x}}$ [36]. In Appendix B we generalise this to the isotropic radial modes $\Phi_{\beta}^0(\chi_{\mathbf{y}})$ associated with general FRW spacetimes:

$$\Phi_{\beta}^0(\chi_{\mathbf{y}}) = \sum_{n=0}^{\infty} (2n+1) \Phi_{\beta}^n(\chi_{\mathbf{x}}) \Phi_{\beta}^n(\chi_{\mathbf{x}+\mathbf{y}}) P_n(\cos \sigma) . \quad (22)$$

The result (22) is valid for all \mathbf{y} . We are, however, principally interested in the superhorizon limit, under which this relation simplifies greatly. In the superhorizon limit, it is easy to show that $\Phi_{\beta}^l(\chi) \propto (\beta\chi)^l$. Also, \mathbf{y} superhorizon ($\beta\chi_{\mathbf{y}} \ll 1$) implies that at least one of $-\mathbf{x}$, $\mathbf{x}+\mathbf{y}$ is superhorizon also, for any given \mathbf{x} . Hence we can make use of the first order approximation only:

$$\Phi_{\beta}^0(\chi_{\mathbf{y}}) \approx \Phi_{\beta}^0(\chi_{\mathbf{x}}) \Phi_{\beta}^0(\chi_{\mathbf{x}+\mathbf{y}}) , \quad (23)$$

for \mathbf{y} superhorizon. Thus the lowest radial eigen-mode plays a role analogous to that of the exponential in Fourier theory.

While Appendix B provides a formal proof of the full expansion (22), here we just obtain (23). Consider an eigenfunction expansion of $\Phi_{\beta}^0(\chi_{\mathbf{u}+\mathbf{v}})$ in terms of $\Phi_{\beta}^0(\chi_{\mathbf{u}})$, after translation to $\mathbf{u}+\mathbf{v}$, which is given by

$$\begin{aligned} \Phi_{\beta}^0(\chi_{\mathbf{u}+\mathbf{v}}) &= \sum_{l,m} a_{lm}^{\beta}(\mathbf{u}) \Phi_{\beta}^l(\chi_{\mathbf{v}}) Y_{lm}(\theta_{\mathbf{v}}, \phi_{\mathbf{v}}) , \\ a_{lm}^{\beta}(\mathbf{u}) &= \frac{1}{N_l} \int dV_{\mathbf{v}} \Phi_{\beta}^0(\chi_{\mathbf{u}+\mathbf{v}}) \Phi_{\beta}^l(\chi_{\mathbf{v}}) Y_{lm}(\theta_{\mathbf{v}}, \phi_{\mathbf{v}}) , \end{aligned}$$

where $N_l = \sqrt{4\pi} \int dV_{\mathbf{v}} [\Phi_{\beta}^l(\chi_{\mathbf{v}}) Y_{lm}(\theta_{\mathbf{v}}, \phi_{\mathbf{v}})]^2$. We shall demonstrate that $a_{00}^{\beta}(\chi_{\mathbf{u}}) = \sqrt{4\pi} \Phi_{\beta}^0(\chi_{\mathbf{u}})$, so that (for $\mathbf{u} = -\mathbf{x}$ and $\mathbf{v} = \mathbf{x}+\mathbf{y}$) eqn (23) is confirmed, on superhorizon scales where this coefficient carries the leading order behaviour. Here, we give only the open universe argument, that for the $K \geq 0$ cosmologies being similar. Substituting for the volume element, for the $l = 0$ spherical harmonic, and for each of the two radial eigenfunctions using (A4), yields

$$a_{00}^{\beta}(\chi_{\mathbf{u}+\mathbf{v}}) = \frac{\sqrt{\pi}}{|K|^{3/2} \beta^2 N_0} \int_0^{\infty} d\chi_{\mathbf{v}} \sinh \chi_{\mathbf{v}} \sin(\beta \chi_{\mathbf{v}}) \int_0^{\pi} d\theta \sin \theta \frac{\sin(\beta \chi_{\mathbf{u}+\mathbf{v}})}{\sinh \chi_{\mathbf{u}+\mathbf{v}}} .$$

Changing variables from θ to $\xi \equiv \chi_{\mathbf{u}+\mathbf{v}}$ using (B2), we obtain the relation $\sin \theta d\theta = \sinh \xi d\xi / \sinh \chi_{\mathbf{v}} \sinh \chi_{\mathbf{u}}$, and calculate the inner integral to be $\frac{2 \sin(\beta \chi_{\mathbf{v}}) \sin(\beta \chi_{\mathbf{u}+\mathbf{v}})}{\beta \sinh \chi_{\mathbf{v}} \sinh \chi_{\mathbf{u}}}$. Hence,

$$\begin{aligned}
a_{00}^\beta(\chi_{\mathbf{v}}) &= \frac{2\sqrt{\pi}}{|K|^{3/2}\beta^2 N_0} \frac{\sin(\beta\chi_{\mathbf{u}})}{\beta\sinh\chi_{\mathbf{u}}} \int_0^\infty d\chi_{\mathbf{v}} \sin^2\beta\chi_{\mathbf{v}} = \sqrt{4\pi} \frac{\sin\beta\chi_{\mathbf{u}}}{\beta\sinh\chi_{\mathbf{u}}} \\
&= \sqrt{4\pi} \Phi_\beta^0(\chi_{\mathbf{u}}) ,
\end{aligned} \tag{24}$$

where we have used the normalisation $N_0 = (|K|^{3/2}\beta^2)^{-1} \int_0^\infty d\chi \sin^2\beta\chi$ in the last equality. Observe that, formally, we should evaluate N_0 , and effect the cancellation in (24), in terms of a limit of a volume larger than all length scales of interest.

B. Integration by parts

Substituting (23) in (13), transforming the inner integration variable to $\mathbf{x} + \mathbf{y}$, and noting that the integrals are over all space so that $dV_{\mathbf{x}+\mathbf{y}} = dV_{\mathbf{y}}$, we see that the $l = 0$ mode is given by the double integral in \mathbf{x} and $\mathbf{x} + \mathbf{y}$:

$$\begin{aligned}
\langle \tau_{j|i}^i(\mathbf{x}) \tau_{q|p}^p(\mathbf{x} + \mathbf{y}) \rangle_{00}(k) &= \frac{1}{\sqrt{2\pi^2}} \int dV_{\mathbf{x}} \tau_{j|i}^i(\mathbf{x}) \Phi_\beta^0(\chi_{\mathbf{x}}) \\
&\times \int dV_{\mathbf{x}+\mathbf{y}} \tau_{j|i}^i(\mathbf{x} + \mathbf{y}) \Phi_\beta^0(\chi_{\mathbf{x}+\mathbf{y}}) .
\end{aligned} \tag{25}$$

In order to evaluate the integrals in (25) we make use of a generalised form of Gauss's Law [35]

$$\int_{G^3} W_a{}^{;a} dV = \int_{G^2} W_a dS^a , \tag{26}$$

where $dV = \sqrt{-g^{(3)}} dx^1 dx^2 dx^3$ and $dS^a = g^{(3)ab} dS_a$, $dS_a = (1/2!) \epsilon_{am_1 m_2} d_1 x^{m_1} d_2 x^{m_2}$. Here, we may choose the 3-dimensional vector components $d_i x^{m_i}$ such that $d_i x^{m_i}$ appears as the i -th component while the rest vanish (for instance, $d_2 x^{m_2} = (0, dx^{m_2}, 0)$). As we are on a constant time hypersurface, the index a runs over 1, 2, 3 only, as do the m_i .

Now, if we replace W^a with fW^a in (26) where f is some scalar quantity (so that $f_{;a} = f_{,a}$) and apply the chain rule, we obtain the result

$$\begin{aligned}
\int_{G^3} dV f W_a{}^{;a} &= \int_{G^3} dV (f W_a)^{;a} - \int_{G^3} dV f^{;a} W_a \\
&= \int_{G^2} dS^a (f W_a) - \int_{G^3} dV f^{;a} W_a .
\end{aligned} \tag{27}$$

This is the formal expression of spatial integration by parts on a constant time hypersurface, and a similar result holds with the covariant and contravariant indices interchanged. We shall exploit this result twice in each of the two integrals in (25).

We shall proceed in some detail for the integral over \mathbf{x} . The $\mathbf{x} + \mathbf{y}$ integral is similar. Since the τ_j^i transform as tensors, setting $f = \Phi_\beta^0(\chi_{\mathbf{x}})$ and $W_j = \tau_{j|i}^i$ so that fW_j is a vector, we find that

$$\begin{aligned}
\int dV_{\mathbf{x}} \Phi_\beta^0(\chi_{\mathbf{x}}) \tau_{j|i}^i &= \int dV_{\mathbf{x}} \Phi_\beta^0(\chi_{\mathbf{x}}) W_j{}^{;j} \\
&= \int dV_{\mathbf{x}} [\Phi_\beta^0(\chi_{\mathbf{x}}) W_j]{}^{;j} - \int dV_{\mathbf{x}} [\Phi_\beta^0(\chi_{\mathbf{x}})]^{;j} W_j
\end{aligned}$$

$$\begin{aligned}
&= \int d f_{\mathbf{x}}^j \left[\Phi_{\beta}^0(\chi_{\mathbf{x}}) W_j \right] - \int d V_{\mathbf{x}} \left[\Phi_{\beta}^0(\chi_{\mathbf{x}}) \right]^{,j} W_j \\
&= - \int d V_{\mathbf{x}} \left[\Phi_{\beta}^0(\chi_{\mathbf{x}}) \right]^{,j} \tau_{j|i}^i, \tag{28}
\end{aligned}$$

where the third equality is obtained as in (27), and the fourth equality is obtained by substituting for W_j , and assuming that the surface integral term vanishes. Explicitly, the surface integrals vanish for all but the angular integral over the radial surface at infinity, by symmetry, and periodicity. This remaining surface integral may be also be set to zero on physical grounds by using the conservation equations (6) to obtain

$$\lim_{R \rightarrow \infty} \Phi_{\beta}^0(R) \int d\Omega \left[\tau_{j|i}^i \right]_R = \lim_{R \rightarrow \infty} \Phi_{\beta}^0(R) \int d\Omega \left[\dot{\tau}_j^0 \right]_R,$$

and by noting that beyond the horizon the functions $[\Phi_{\beta}^0 \dot{\tau}_j^0]_R$ (although possibly non-zero) should not make a net contribution to the correlator. Physically, we may think of this as an expression of the fact that we do not expect there to be any net bulk momentum transfers across a radial surface on the very largest scales, since τ_0^j is proportional to a well-defined momentum [17]. This is a much less stringent requirement than that previously imposed (*i.e.* that the functions themselves go to zero beyond the horizon [3,19]) which is not expected to be true in most physical situations of interest. Our physically motivated and much weaker assumption thus allows for a more complete analysis of the superhorizon behaviour.

Repeating the analysis, we find that

$$\int d V_{\mathbf{x}} \Phi_{\beta}^0(\chi_{\mathbf{x}}) \tau_{j|i}^i{}^{,j} = + \int d V_{\mathbf{x}} \left\{ \left[\Phi_{\beta}^0(\chi_{\mathbf{x}}) \right]^{,j} \right\}_{|i} \tau_{j|i}^i, \tag{29}$$

where we have once again ignored the surface terms. Explicitly, the only non-trivial surface term is again the radial surface at infinity:

$$\lim_{R \rightarrow \infty} \frac{\partial \Phi_{\beta}^0(R)}{\partial R} \int d\Omega_{\mathbf{x}} \left[\tau_1^i \right]_R$$

and, this time, we expect there to be no net anisotropic shears across the radial surface on the largest scales, so that this surface term averages to zero and may be ignored. Hence, using the connection coefficients, as well as the derivative (A7) and recursion (A6) relations for the Φ_l^0 , we see that

$$\int d V_{\mathbf{x}} \Phi_{\beta}^0(\chi_{\mathbf{x}}) \tau_{j|i}^i{}^{,j} = - \int d V_{\mathbf{x}} \Phi_{\beta}^0 f_K \left(\tau_2^2 + \tau_3^3 \right) + \int d V_{\mathbf{x}} \Phi_{\beta}^0(\chi) \left[-\tilde{k}^2 + 2f_K \right] \tau_1^1, \tag{30}$$

where

$$f_K(\beta, \chi) \equiv \cot_K^2 \chi - \beta \cot_K \cot(\beta \chi), \tag{31}$$

and $\cot_K \chi = \cos_K \chi / \sin_K \chi$ where $\cos_K \chi = \{\cosh \chi, \chi, \cos \chi\}$ for $\{K < 0, K = 0, K > 0\}$.

Proceeding in a similar fashion for the $\mathbf{x} + \mathbf{y}$ integral, multiplying to obtain four double integrals, and then reversing the steps that led to (25) — *i.e.* transforming back to \mathbf{y} as

the integration variable in the inner integral, and appealing to (23) — we may write the 00-mode as

$$\begin{aligned}
\langle \tau_{j|i}^i \tau_{q|p}^p \rangle_{00}(\beta) = & \\
& \langle [-\tilde{k}^2 + 2f_K(\beta, \chi_{\mathbf{x}})] \tau_1^1(\mathbf{x}) [-\tilde{k}^2 + 2f_K(\beta, \chi_{\mathbf{x}+\mathbf{y}})] \tau_1^1(\mathbf{x} + \mathbf{y}) \rangle_{00}(\beta) \\
& + \langle [-\tilde{k}^2 + 2f_K(\beta, \chi_{\mathbf{x}})] \tau_1^1(\mathbf{x}) [-f_K(\beta, \chi_{\mathbf{x}+\mathbf{y}})] (\tau_2^2 + \tau_3^3)(\mathbf{x} + \mathbf{y}) \rangle_{00}(\beta) \\
& + \langle -f_K(\beta, \chi_{\mathbf{x}}) (\tau_2^2 + \tau_3^3)(\mathbf{x}) [-\tilde{k}^2 + 2f_K(\beta, \chi_{\mathbf{x}+\mathbf{y}})] \tau_1^1(\mathbf{x} + \mathbf{y}) \rangle_{00}(\beta) \\
& + \langle -f_K(\beta, \chi_{\mathbf{x}}) (\tau_2^2 + \tau_3^3)(\mathbf{x}) [-f_K(\beta, \chi_{\mathbf{x}+\mathbf{y}})] (\tau_2^2 + \tau_3^3)(\mathbf{x} + \mathbf{y}) \rangle_{00}(\beta) .
\end{aligned} \tag{32}$$

Each of the four correlators on the right hand side of (32) is equal to a double integral similar in form (though, clearly not in integrand) to (25). As remarked earlier, for \mathbf{y} superhorizon, at least one of \mathbf{x} , $\mathbf{x} + \mathbf{y}$ is superhorizon also so that we may take the integral over that variable to be finite. Effectively, then, we have obtained the transformed two-point correlators of the contracted derivatives of the pseudo-stresses—and hence the pseudo-energy—in terms of functions of k times the transformed pseudo-stresses.

C. Flat space

For the flat space ($K = 0$) case, on sufficiently superhorizon scales ($kr \ll 1$) we may take the limit

$$f_{K=0} = \frac{1}{r^2} - \frac{k \cot(kr)}{r} \approx \frac{k^2}{3} \quad \text{for } kr \ll 1 ,$$

to show that the pre-factors $(-k^2 + 2f_{K=0})^2$, $-f_{K=0}(-k^2 + 2f_{K=0})$, $f_{K=0}^2$ are functions of k only, in the limit $kr \rightarrow 0$. Hence, we may pull these factors out of the (implicit) integrals in (32) and, taking the limit to infinity, we obtain

$$\begin{aligned}
\langle \tau_{j|i}^i \tau_{q|p}^p \rangle_{00}(k) = & \frac{k^4}{9} \left[\langle \tau_1^1 \tau_1^1 \rangle_{00}(k) + \langle \tau_1^1 (\tau_2^2 + \tau_3^3) \rangle_{00}(k) \right. \\
& \left. + \langle (\tau_2^2 + \tau_3^3) \tau_1^1 \rangle_{00}(k) + \langle (\tau_2^2 + \tau_3^3) (\tau_2^2 + \tau_3^3) \rangle_{00}(k) \right] ,
\end{aligned} \tag{33}$$

where the correlators on the right hand side of (33) are known (see §IV B) to be proportional to k^0 in this limit. Hence, appealing to the conservation equations (6) and noting that time differentiation does not affect β -dependence in our eigenfunction expansions, we regain the familiar k^4 power spectrum, as the spectrum for the 00-mode

$$\langle \tau_0^0(\mathbf{x}) \tau_0^0(\mathbf{x} + \mathbf{y}) \rangle_{00}(k) \propto k^4 . \tag{34}$$

D. Open universe

In this case, there are three regimes of interest: supercurvature, sub-curvature, and the case in which χ is near the curvature scale. On superhorizon, supercurvature scales, $\chi \gg 1$, $\beta\chi \ll 1$ which implies that $\beta \ll 1/\chi \ll 1$. Hence, we have that $\coth\chi \approx 1$ and $\tilde{k} \approx 1$, where these approximations are exact in this limit, and about which we have the expansion

$$f_{K<0}(\beta, \chi) = 1 - \beta \cot(\beta\chi) = 1 - \frac{1}{\chi} + \frac{\beta^2\chi}{3} \dots = O(1) , \quad \beta\chi \ll 1 , \quad \chi \gg 1 . \quad (35)$$

Substituting (35) into (32), we see that the 00-mode tends to a constant on supercurvature, and superhorizon scales. Explicitly, making the $\tilde{k} \approx 1$ approximation valid on these superhorizon and supercurvature scales, we obtain

$$\begin{aligned} \langle \tau_{j|i}^i \tau_{j|i}^j \rangle_{00}(\beta) &= a_1 \langle \tau_1^1 \tau_1^1 \rangle_{00}(\beta) + a_2 \langle \tau_1^1 (\tau_2^2 + \tau_3^3) \rangle_{00}(\beta) \\ &\quad + a_3 \langle (\tau_2^2 + \tau_3^3) \tau_1^1 \rangle_{00}(\beta) + a_4 \langle (\tau_2^2 + \tau_3^3) (\tau_2^2 + \tau_3^3) \rangle_{00}(\beta) , \\ \text{for } \chi_{\mathbf{x},\mathbf{y}} &\gg 1, \quad \beta\chi_{\mathbf{x},\mathbf{y}} \ll 1 , \end{aligned}$$

where the a_i are constant factors. Recalling the equations of motion (6) for the energy-momentum pseudo-tensor, and that the correlator between two arbitrary functions goes as β^0 , we see that on very supercurvature and very superhorizon scales, the correlators behave as:

$$\langle \tau_0^0 \tau_0^0 \rangle_{00} \propto \beta^0 . \quad (36)$$

Turning now to the other asymptote, we consider the case in which the horizon is very much less than the curvature scale. Then, on superhorizon, and very much subcurvature scales, we regain the familiar k^4 power spectrum, up to a constant factor. Using the series expansions appropriate for $\chi \ll 1$ and $\beta\chi \ll 1$, one easily finds that

$$f_{K<0} = \frac{\beta^2 + 1}{3} = \frac{\tilde{k}^2}{3} , \quad \beta\chi \ll 1 , \chi \ll 1 . \quad (37)$$

This has the same form as the superhorizon limit of $f_{K=0}$. However, one should recall that, while β (for the $K < 0$ case) and k (for $K = 0$) have similar spectra, their meanings are different in the two spacetimes — see equation (11). In particular, we have, for the open universe with $\chi \ll 1$, two cases: $\beta < 1$ and $\beta > 1$. These correspond to mode wavelengths approximately equal to and smaller than the curvature horizon, respectively. In the former case, $f_{K<0} \approx 1/3$ and $\tilde{k} \approx 1$ so that we have the result (36) once more. In the latter, $f_{K<0} \approx \beta^2/3$, for large β , in which case we also have that $\tilde{k}^2 \approx \beta^2$. Then, recalling the normalisation of k , we have the local (subcurvature) superhorizon limit:

$$f_{K<0}(\chi, \beta) \approx \frac{1}{|K|} f_{K=0}(\tilde{k}, r) .$$

Recalling the conservation equations (6), this means that

$$\langle \tau_0^0(\mathbf{x}) \tau_0^0(\mathbf{x} + \mathbf{y}) \rangle_{00}(\beta) \propto \tilde{k}^4 = \frac{k^4}{|K|^2} \approx \beta^4, \quad \chi_{\mathbf{x},\mathbf{y}} \ll 1, \quad \beta \chi_{\mathbf{x},\mathbf{y}} \ll 1, \quad (38)$$

where k is the flat space Fourier wavenumber.

Having dealt with both the extreme cases of local flatness and curvature domination, there remains the interesting question of the behaviour near the curvature scale. We therefore would like to understand how the k^4 superhorizon behaviour on sub-curvature scales becomes $\beta^0 = k^0$ on very super-curvature scales, and the ranges within which these approximations are valid.

Away from the asymptotes dealt with above, $f_{K<0}$ is a function of the radial coordinate, and therefore is integrated over (twice) in equation (32). This means that we may only truly understand the effect of the factors in this equation by explicitly calculating the integral, which requires that we specify the pseudo-stresses. As a first approximation, however, we may proceed to obtain some understanding of the superhorizon behaviour of generic causal perturbations by assuming that the corrections are small enough for us to make the approximation

$$\langle \tau_0^0(\mathbf{x}) \tau_0^0(\mathbf{x} + \mathbf{y}) \rangle_{00} \approx (P1 + P2 + P3 + P4)A_{00}, \quad (39)$$

where

$$\begin{aligned} P1 &= (-\tilde{k}^2 + 2f_{K<0})^2, & P2 &= -f_{K<0}(-\tilde{k}^2 + 2f_{K<0}), \\ P3 &= -(-\tilde{k}^2 + 2f_{K<0})f_{K<0}, & P4 &= f_{K<0}^2, \end{aligned}$$

for some given χ , and where we have assumed that the correlators of the τ_j^i are well modelled by window functions of the form used in §IV A. This assumption amounts to ignoring long range correlations, since we assume that $\chi_{\mathbf{x}} \approx \chi_{\mathbf{x}+\mathbf{y}}$. Hence this approximation is valid for $\chi_{\mathbf{y}}$ small, that is, in the very early universe.

We provide surface plots for these pre-factors in Figures 3–5, from which we can clearly see how our approximations are valid only for $\chi < 0.5$, and break down as we move into the large χ and large β regime, doing so much earlier than in the $K = 0$ scenario. This is a geometric effect due to the fact that the majority of the volume in a hyperbolic space is located near the radial surface, so that a large ‘hyper-sphere’ approximates a ‘hyper-shell’. This effect is most noticeable on supercurvature scales. However, one can calculate that the ratio of hyperbolic volume to surface area rapidly diverges from that for a spherical geometry, beginning at around $\chi \sim 0.5$. On the other hand, the steep slope in the prefactor P_1 for large β and $\chi < 0.4$ is due to the appearance of a constant (curvature) term in the (repeated) \tilde{k} factors in this prefactor. However, as χ increases, so does $f_{K<0}$, first cancelling, and then overriding this effect. Along the β axis in each plot (*i.e.* for $\chi \sim 0$), we see that all the pre-factors go as $k^4/9$ for $K = 0$ and as $(\beta^2 + 1)^2/9$ for $K = -1$. Along the χ axis ($\beta \sim 0$), all the pre-factors vanish for $K = 0$. However, for the $K = -1$ case, this is not the case, reflecting the small constant (curvature) term.

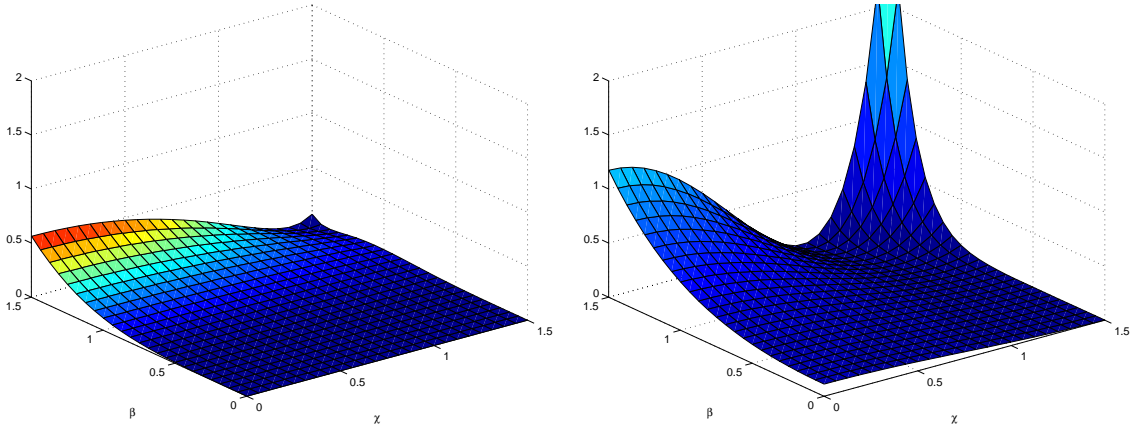


FIG. 3. P1: $K = 0$ (LHS), and $K = -1$ (RHS)

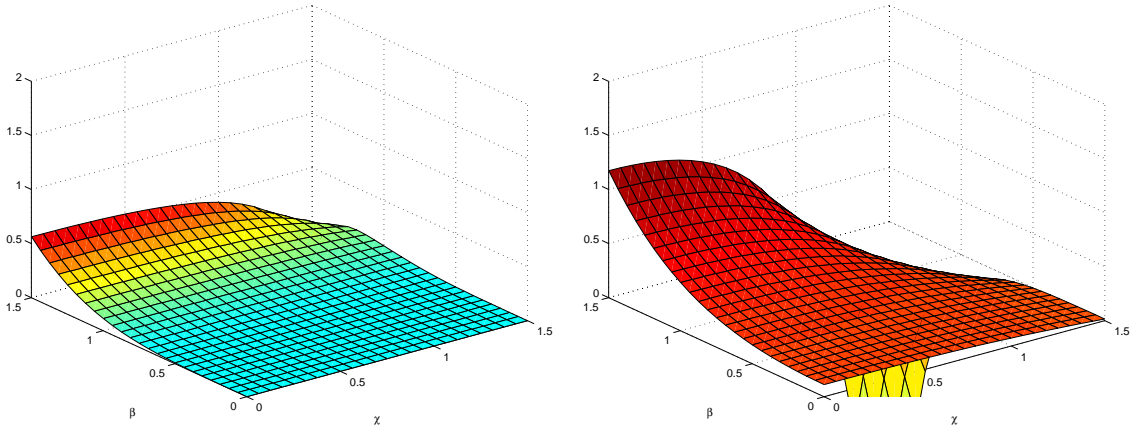


FIG. 4. $P2 = P3$ (in this limit): $K = 0$ (LHS), $K = -1$ (RHS)

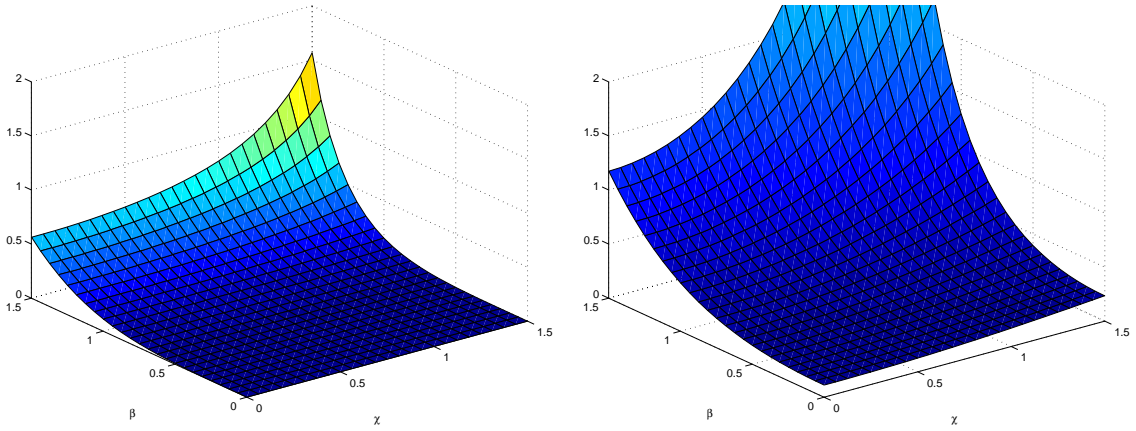


FIG. 5. P4: $K = 0$ (LHS), $K = -1$ (RHS)

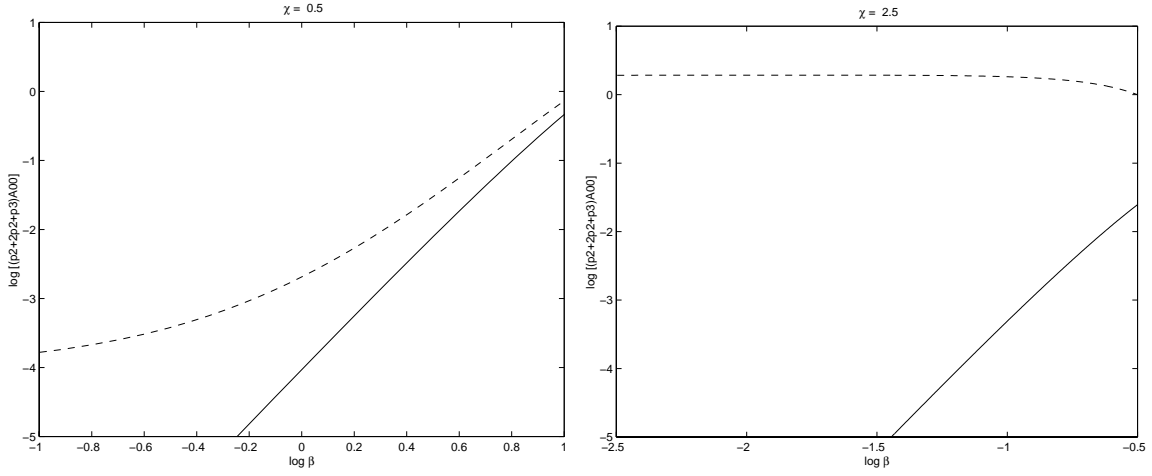


FIG. 6. Log $(P1 + 2P2 + P3)A00$ for $K = 0$ (-) and $K = -1$ (- -)

Computing $\langle \tau_0^0 \tau_0^0 \rangle$ using the approximation (39) we find the curves plotted in Figure 6. These show clearly how the k^4 behaviour on superhorizon scales is flattened out, for small β , by the intrinsic curvature. We note that this does not mean that we expect the real space correlators to similarly go to a constant on these scales because, of course, they vanish by causality. Integrating (39) over all β , with the kernel Φ_β^0 , the constant term is suppressed by the measure $d\mu(\beta) = \beta^2 d\beta$. Similarly, calculating the variance

$$\sigma^2 = \int_0^\infty d\mu(\beta) \langle \tau_0^0 \tau_0^0 \rangle_{00}(\beta) = \int_0^1 d\mu(\beta) \langle \tau_0^0 \tau_0^0 \rangle_{00}(\beta) + \int_1^\infty d\mu(\beta) \langle \tau_0^0 \tau_0^0 \rangle_{00}(\beta),$$

and breaking the integral over all β into supercurvature and subcurvature parts, we see that the second integral is procedurally the same as for the flat space case, for both sub- and superhorizon scales, while the first (containing the constant power spectrum contribution) is suppressed by the β^2 factor in the measure.

E. Closed universe

For the closed universe we have the prefactors determined by

$$f_{K>0}(\beta, \chi) = \cot^2 \chi - \beta \cot \chi \cot(\beta \chi). \quad (40)$$

The difference between the open and closed universe cases is quantified by the differences between the behaviour of the function (40) and that of $f_{K<0}$. To begin with, the eigenspectrum for β is discrete, and the lowest non-gauge mode is $\beta = 3$ so that the superhorizon limit $\beta \chi_H \ll 1$ can only be attained for $\chi_H \ll 1/3$. Of course, for $\chi_H \gg 1$ we encounter the periodic properties the closed universe: one cannot have modes that are both superhorizon and supercurvature.

If we take $\chi \ll 1$ and $\beta \chi \ll 1$, then we consider the subcurvature yet superhorizon case. Here, we note that $\cot \chi \approx 1/\chi - \chi/3$, so we have

$$f_{K>0} = \frac{\beta^2 - 1}{3} = \frac{\tilde{k}^2}{3}, \quad \beta \chi \ll 1, \quad \chi \ll 1, \quad (41)$$

and we regain the flat space results in a similar fashion to the open universe case. Hence, an analysis similar to that for the open universe shows that we once again find the local limit

$$\langle \tau_0^0(\mathbf{x}) \tau_0^0(\mathbf{x} + \mathbf{y}) \rangle \propto \frac{\tilde{k}^4}{9} \quad (42)$$

where $\tilde{k}^4 = (\beta^2 - 1)^2 \approx k^4/|K|^2$ for large β , with k being the flat space Fourier wavenumber.

VI. DISCUSSION

It would be of great physical interest if the consequences of this superhorizon behaviour were within the range of observational searches because it would test ‘active’ theories for structure formation, and possibly probe the curvature of the universe. The cosmic microwave sky may provide the best prospect, however, for the CMB temperature spectrum the superhorizon ‘fall-off’ expected on the very largest scales will be confused by cosmic variance. The most likely signature would be through the CMB polarization spectrum because it represents a primordial snapshot of the universe at recombination which includes about 10^5 different horizon volumes. The lack of spatial correlations on scales corresponding to the horizon size at that time (about 2°), and the fall-off in their transformed counterparts, should be observationally testable. Previous analysis of this possibility [8], however, has only considered scalar modes, so we comment on the possible impact of vector and tensor modes here. We leave discussion of the observational imprint of this superhorizon behaviour for a future publication [30], concentrating instead on the consistent implementation of initial conditions for numerical simulations.

A. SVT decomposition and superhorizon coherence

The decomposition of a tensor perturbation into its scalar, vector and tensor parts is not without its subtleties [12], since the projection operators yielding the scalar, vector and tensor parts are not, in general, analytic. The spatial distribution of the separate geometric parts may therefore be significantly different from the physical distribution of the entire object, so that the decomposed parts of a causal tensor object can manifest acausal features. The transformed variables will then in turn exhibit non-analytic behaviour [5], complicating the analysis of their superhorizon behaviour. However, causality is merely hidden, and, since the arguments presented in this paper for the (scalar) energy density perturbation are (trivially) phrased in terms of full tensors, they escape this difficulty.

On the other hand, it is common to employ the assumption of coherence, in which case one regains analyticity [5], all the decomposed parts of the perturbation are causal, and one may obtain the power spectra by simple power counting. This allows us to write down the auto-correlators for both the scalar and the vector parts of the pseudo-tensor τ^μ_ν , under the influence of energy-momentum conservation.

If we decompose the pseudo-tensor τ^μ_ν as in (A11), we may write the energy-momentum equations (6) as

$$\begin{aligned}
\tau_{IV} &= \frac{\dot{\tau}_S}{k} , & k\dot{\tau}_{IV} &= 2k^2 \left[\tau_L + 2 \left(\frac{K}{k^2} - \frac{1}{3} \right) \tau_T \right] = \ddot{\tau}_S , \\
\tau_{IT}^{(\pm 1)} &= -2k[k^2 - 2K]^{-1} \dot{\tau}_V^{(\pm 1)} = \frac{\ddot{h}_V^{(\pm 1)}}{\kappa} ,
\end{aligned} \tag{43}$$

where we have used the first equation to obtain the final equality in the second. Assuming coherence, we find that the scalar (τ_S) and vector (τ_{IV} , τ_V) degrees of freedom have auto-correlators that go as

$$\begin{aligned}
\langle |\tau_S|^2 \rangle(\beta) &\propto \tilde{k}^2 \langle |\tau_{IV}|^2 \rangle(\beta) , \\
\langle |\tau_{IV}|^2 \rangle(\beta) &\propto \tilde{k}^2 \left(4 \langle |\tau_L|^2 \rangle(\beta) - \frac{16}{3} \langle \tau_L \tau_T^* \rangle(\beta) + \frac{16}{9} \langle |\tau_T|^2 \rangle(\beta) \right) \\
&\quad + 16K \left(\langle |\tau_L|^2 \rangle - \frac{2}{3} \langle |\tau_T|^2 \rangle(\beta) \right) + 16 \frac{K^2}{\tilde{k}^2} \langle |\tau_T|^2 \rangle(\beta) , \\
\langle |\tau_V^{(\pm 1)}|^2 \rangle(\beta) &\propto \left(\frac{1}{4} \tilde{k}^2 - K + \frac{K^2}{\tilde{k}^2} \right) \langle |\tau_{IT}^{(\pm 1)}|^2 \rangle(\beta) ,
\end{aligned} \tag{44}$$

on superhorizon scales. Moreover, the correlators for τ_L , τ_T , and τ_{IT} have white noise superhorizon behaviour (refer to §IV B and [5]) so we see that in the local limit ($\chi \ll 1$, $\beta \gg 1$) we regain the flat space results, with the scalar power spectrum $\propto \tilde{k}^4$, while those of the vectors is proportional to \tilde{k}^2 . In the supercurvature limit, they all tend to a constant. Note that the correlators in (44) do not diverge as $\beta \rightarrow 0$, since \tilde{k} tends to 1 in this limit.

The results (44) are more general than one might suppose upon first inspection, as an arbitrary decoherent source may be written as a ‘decoherent sum’ of coherent sources, each of which will then be amenable to the above arguments [21]. Moreover, we believe that it is plausible to expect approximate coherence on the very largest scales, since the eigenfunctions used to diagonalise the ‘decoherent sum’ are likely to either all tend to white noise on these scales, or to be dominated by just one element in the sum — see also ref. [30]. This notion has been ventured before as an hypothesis [31] for scaling sources, and is supported by both numerical results [32], and by the observation that, for causal sources, modes with wavelengths much larger than the horizon cannot be significantly out of phase with one another. We shall consider this point in more detail elsewhere [30].

B. Matching conditions and initial conditions

Finally, we wish to propose a set of initial conditions which would find useful application in numerical simulations of causal perturbation models, such as cosmic defects. We shall assume, as is usual, that we are dealing primarily with the evolution of transformed quantities. By considered an instantaneous phase transition early in the universe as a first approximation to a model for cosmic defects “switching on”, and employing a gauge in which constant energy and constant time surfaces coincide, matching conditions can be used to show that there exists an entire class of objects which are continuous across the transition and are related by gauge transformation to our pseudo-tensor components.

In a previous paper (ref. [17]), we used the notion of compensation together with a particular gauge specification to remove this redundancy such that the τ_S (pseudo-energy) and $\tau_V^{(\pm 1)}$ (divergenceless vector) components of our generalized pseudo-tensor have this property:

$$[\tau_S]_{\pm} = 0, \quad [\tau_V^{\pm 1}]_{\pm} = 0.$$

While we emphasise that this matching depends on a special gauge choice, it is one which should be widely applicable in many physical situations, for example, a defect-forming transition [17].

For a universe which was unperturbed (and hence homogeneous and isotropic) prior to the transition, we may then take $\tau_S = 0 = \tau_V^{(\pm 1)}$ as natural initial conditions. This result is consistent on all scales and is especially appropriate for suppressed superhorizon modes. The results of §V confirm this conclusion for the density perturbation tracked by τ_S , and the previous section argues that this conclusion may be extended also to the ‘induced vector’ mode τ_{IV} and the true vector modes $\tau_V^{\pm 1}$. Hence, if one wishes to set initial conditions for perturbations initially outside the horizon, one may consistently set both the pseudo-energy τ_0^0 and the pseudo-momentum τ_i^0 to zero. If one wishes to include long wavelength modes that are affected by curvature, then one can proceed in a similar fashion. Although the transformed correlators become constant on these scales, at early times they are sufficiently outside the horizon for their amplitude to be strongly suppressed. Therefore, the matching conditions we propose as initial conditions will also be applicable and self-consistent for a curved universe.

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APPENDIX A: EIGENFUNCTIONS OF THE LAPLACIAN

The scalar eigenfunctions are given by solutions to (10) with $|m| = 0$ (*c.f.*, with a slightly different normalisation, ref. [15]):

$$\zeta_{\beta lm}(\mathbf{x}) = \sqrt{\frac{2}{\pi}} \Phi_l^{\beta}(y) Y_l^m(\theta, \phi). \quad (\text{A1})$$

Here the angular dependence is contained in the spherical harmonics which are given by $Y_l^m(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} P_l^m(\cos\theta) e^{im\phi}$ (where P_l^m is an associated Legendre function), while the radial function Φ_l^{β} is

$$\Phi_\beta^l = \begin{cases} \left(\frac{\pi N_\beta^l}{2\beta^2 \sinh \chi} \right)^{\frac{1}{2}} P_{-\frac{1}{2}+l}^{-\frac{1}{2}-l}(\cosh \chi) & K < 0 \\ j_l(\beta \chi) = j_l(kr) & K = 0 \\ \left(\frac{\pi M_\beta^l}{2\beta^2 \sin \chi} \right)^{\frac{1}{2}} P_{-\frac{1}{2}+\beta}^{-\frac{1}{2}-l}(\cos \chi) & K > 0 \end{cases}, \quad (\text{A2})$$

with $N_\beta^l = \Pi_{n=0}^l(\beta^2 + n^2)$ and $M_\beta^l = \Pi_{n=0}^l(\beta^2 - n^2)$. By Taylor expanding about $\beta \chi$ small, we may show that $\Phi_\beta^l \sim j_l(kr)$ in the limit $y \rightarrow 0$, $\beta \rightarrow \infty$.

The basis $\{\zeta_{\beta lm}\}$ is orthonormal as well as complete, and is normalised according to

$$\int_S dV \zeta_{\beta' l' m'}^* \zeta_{\beta l m} = \delta(\beta', \beta) \delta_{l' l} \delta_{m' m}, \quad (\text{A3})$$

where $\delta(\beta' \beta)$ is the delta function with respect to the measure $\mu(\beta)$:

$$\begin{aligned} \int d\mu(\beta) &= \int_0^\infty \beta^2 d\beta, & \delta(\beta, \beta') &= \frac{1}{\beta^2} \delta(\beta - \beta'), & \text{for } K \leq 0, \\ \int d\mu(\beta) &= \sum_{\beta=3}^\infty \beta^2, & \delta(\beta, \beta') &= \frac{1}{\beta^2} \delta_{\beta \beta'}, & \text{for } K > 0. \end{aligned}$$

The $l = 0$ and $l = 1$ radial modes are given [15] by the exact formulae (rewritten for a continuous range of K values):

$$\Phi_\beta^0 = \frac{\sin(\beta \chi)}{\beta} \times \begin{cases} \frac{1}{\sinh \chi} & K < 0 \\ \frac{1}{\chi} & K = 0 \\ \frac{1}{\sin \chi} & K > 0 \end{cases}, \quad (\text{A4})$$

$$\Phi_\beta^1 = \Phi_\beta^0 \times \begin{cases} (\beta^2 + 1)^{-\frac{1}{2}} [\coth \chi - \beta \cot(\beta \chi)] & K < 0 \\ \frac{1}{\beta} \left[\frac{1}{\chi} - \beta \cot(\beta \chi) \right] & K = 0 \\ (\beta^2 - 1)^{-\frac{1}{2}} [\cot \chi - \beta \cot(\beta \chi)] & K > 0 \end{cases}. \quad (\text{A5})$$

The radial function Φ_β^l may be recursively calculated using the exact formulae (A4), (A5) and the relations:

$$\Phi_\beta^l = \begin{cases} (\beta^2 + l^2)^{-\frac{1}{2}} \left[(2l - 1)(\coth \chi) \Phi_\beta^{l-1} - [\beta^2 + (l - 1)^2]^{\frac{1}{2}} \Phi_\beta^{l-2} \right] & K < 0 \\ \frac{1}{\beta} \left[(2l - 1) \frac{1}{\chi} \Phi_\beta^{l-1} - \beta \Phi_\beta^{l-2} \right] & K = 0 \\ (\beta^2 - l^2)^{-\frac{1}{2}} \left[(2l - 1)(\cot \chi) \Phi_\beta^{l-1} - [\beta^2 - (l - 1)^2]^{\frac{1}{2}} \Phi_\beta^{l-2} \right] & K > 0 \end{cases}, \quad (\text{A6})$$

while the derivative with respect to the radial coordinate may be found using

$$\frac{d}{d\chi} \Phi_\beta^l = \begin{cases} l(\coth \chi) \Phi_\beta^l - [\beta^2 + (l + 1)^2]^{\frac{1}{2}} \Phi_\beta^{l+1} & K < 0 \\ \frac{l}{\chi} \Phi_\beta^l - \beta \Phi_\beta^{l+1} & K = 0 \\ l(\cot \chi) \Phi_\beta^l - [\beta^2 - (l + 1)^2]^{\frac{1}{2}} \Phi_\beta^{l+1} & K > 0 \end{cases}. \quad (\text{A7})$$

Using (A7) and (A6) for Φ_β^1 and Φ_β^2 respectively, one may obtain the relations:

$$\begin{aligned} \frac{d}{d\chi} \left[\frac{\sinh^2 \chi}{\sqrt{\beta^2 + 1}} \Phi_\beta^1(\chi) \right] &= \sinh^2 \chi \Phi_\beta^0(\chi) & K < 0 \\ \frac{d}{d\chi} \left[\frac{\chi^2}{\beta} \Phi_\beta^1(\chi) \right] &= \chi^2 \Phi_\beta^0(\chi) & K = 0 \\ \frac{d}{d\chi} \left[\frac{\sin^2 \chi}{\sqrt{\beta^2 - 1}} \Phi_\beta^1(\chi) \right] &= \sin^2 \chi \Phi_\beta^0(\chi) & K > 0 \end{aligned} \quad (\text{A8})$$

The vector eigenfunctions satisfy (10) for $|m| = 1$, subject to the divergenceless condition $D^i Q_i^{(\pm 1)} = 0$, where the spectrum is defined by $\beta^2 = (k^2 + 2K)/|K|$. The radial component $Q_1^{(\pm 1)}$ is given by

$$Q_1^{(1)} = \frac{1}{\beta} [l(l+1)]^{1/2} \Phi_\beta^l Y_{lm} \begin{cases} \frac{1}{\sinh \chi} & K < 0 \\ \frac{1}{\chi} & K = 0 \\ \frac{1}{\sin \chi} & K > 0 \end{cases} \quad (\text{A9})$$

and similarly for $m = -1$, while the remaining (angular) components may be obtained by substituting (A9) into (10). The vector eigenfunctions satisfy normalisation conditions obtained by replacing $\zeta_{\beta'l'm'}^* \zeta_{\beta lm}$ with $\zeta_{i\beta'l'm'}^{*(\pm 1)} \zeta_{\beta lm}^{(\pm 1)}$ in (A3), with $Q_i^{(\pm 1)}(\beta, l, m) = \zeta_{i\beta lm}^{(\pm 1)}$.

The tensor eigenfunctions satisfy (10) for $|m| = 2$, subject to the transverse-traceless ($D^i Q_{ij}^{(2)} = 0 = Q_i^{(2) i}$) and symmetry ($Q_{ij}^{(2)} = Q_{ji}^{(2)}$) constraints. The spectrum is given by $\beta^2 = (k^2 + 3K)/|K|$. The radial-radial component is given by

$$Q_{11}^{(2)} = \left[\frac{(l+2)(l+1)l(l-1)}{2\beta^2(\beta^2 - K)} \right]^{1/2} \Phi_\beta^l Y_{lm} \begin{cases} \frac{1}{\sinh^2 \chi} & K < 0 \\ \frac{1}{\chi^2} & K = 0 \\ \frac{1}{\sin^2 \chi} & K > 0 \end{cases} \quad (\text{A10})$$

and similarly for $m = -2$. The remaining components may be obtained by substituting (A10) in (10) and the constraints. Normalisation has again been achieved by a replacement of the integrand in (A3) by $\zeta_{ij\beta'l'm'}^{*(\pm 2)} \zeta_{\beta lm}^{(\pm 2)}$.

Using this decomposition we may decompose arbitrary scalar, vector and tensor perturbations. For example, the pseudo-tensor τ_ν^μ defined in (5) may be written as:

$$\begin{aligned} \tau_0^0(\mathbf{x}, \tau) &= \int d\mu(\beta) \tau_S Q^{(0)}, \\ \tau_i^0(\mathbf{x}, \tau) &= \int d\mu(\beta) \left[\tau_{IV} Q_i^{(0)} + \tau_V^{(1)} Q_i^{(1)} + \tau_V^{(-1)} Q_i^{(-1)} \right], \\ \tau_j^i(\mathbf{x}, \tau) &= \int d\mu(\beta) \left[2 \left(\tau_L \gamma^i_j Q^{(0)} + \tau_T Q^{(0) i}_j \right) + \tau_{IT}^{(1)} Q^{(1) i}_j + \tau_{IT}^{(-1)} Q^{(-1) i}_j \right. \\ &\quad \left. + \tau_G^{(2)} Q^{(2) i}_j + \tau_G^{(-2)} Q^{(-2) i}_j \right], \end{aligned} \quad (\text{A11})$$

where the coefficients on the right hand side have suppressed dependence on both β and τ . Their explicit form is given in ref. [17]. Here τ_L and τ_T represent two ‘longitudinal’ and ‘transverse’ scalar degrees of freedom, associated with the auxillary tensors $\gamma^i_j Q^{(0)}$ and $Q^{(0) i}_j = k^{-2} Q^{(0) |i}_{|j} + \frac{1}{3} \gamma^i_j Q^{(0)}$ respectively; τ_{IV} is the vector mode ‘induced’ by the auxillary vector $Q_i^{(0)} = -k^{-1} Q_{|i}^{(0)}$; and $\tau_{IT}^{(\pm 1)}$ are the tensor modes ‘induced’ by $Q_{ij}^{(\pm 1)} = -(2k)^{-1} [Q_{i|j}^{(\pm 1)} + Q_{j|i}^{(\pm 1)}]$. As well as the transform over the ‘radial’ coordinate β , there is an implicit sum over indices ℓm which label the spherical harmonics encoding the angular dependence.

APPENDIX B: “ADDITION” RESULTS FOR $\Phi_\beta^\ell(\chi)$

Here we generalise the result (21) to the $K \neq 0$ and $l \neq 0$ cases, and prove that the radial eigenfunctions at \mathbf{u} and \mathbf{v} are related to those at $\mathbf{u} + \mathbf{v}$ by

$$\frac{\Phi_\beta^n(\chi_{\mathbf{u}+\mathbf{v}})}{(\sinh \chi_{\mathbf{u}+\mathbf{v}})^n} = \Omega_\nu(\chi_{\mathbf{u}+\mathbf{v}}) = \sum_{m=0}^{\infty} b_m \frac{\Phi_\beta^{n+m}(\chi_{\mathbf{u}})}{(\sinh \chi_{\mathbf{u}})^n} \frac{\Phi_\beta^{n+m}(\chi_{\mathbf{v}})}{(\sinh \chi_{\mathbf{v}})^n} C_m^\nu(\cos \sigma), \quad (\text{B1})$$

where b_m is independent of $\chi_{\mathbf{u}}$, $\chi_{\mathbf{v}}$ and σ , and may be found by comparing coefficients. For $\mathbf{u} = -\mathbf{x}$ and $\mathbf{v} = \mathbf{x} + \mathbf{y}$ and for the $l = 0$ case, we may appeal to the common normalisation of our radial eigenfunctions (for the $K = 0$ and $K \neq 0$ universes) to obtain (22).

Foch [37] proved this result for the special case $l = 0$, and for a closed universe only. Here we follow the method of Gegenbauer [38] to obtain a more general result for the l -th radial eigenmode of an open universe. The proof for the closed universe is similar.

For curved geometries, the radial coordinates at \mathbf{u} and \mathbf{v} are related to those at $\mathbf{u} + \mathbf{v}$ via

$$\cos_K \chi_{\mathbf{u}+\mathbf{v}} = \cos_K \chi_{\mathbf{u}} \cos_K \chi_{\mathbf{v}} - \frac{K}{|K|} \sin_K \chi_{\mathbf{u}} \sin_K \chi_{\mathbf{v}} \cos \sigma, \quad (\text{B2})$$

where we use $\sin_K \chi$ and $\cos_K \chi$ as previously defined, but only for the $K \neq 0$ cases. Here, σ is the angle between \mathbf{u} and \mathbf{v} (in the hyperplane defined by these vectors), and where we may (without loss of generality) choose \mathbf{u} pointing towards the north pole such that $\sigma = \theta$. For the open (closed) universes, equation (B2) clearly reduces to the flat space result $r_{\mathbf{u}+\mathbf{v}}^2 = r_{\mathbf{u}}^2 + r_{\mathbf{v}}^2 + 2r_{\mathbf{u}}r_{\mathbf{v}}\cos\sigma$, in the local limit and for $|K| \rightarrow 0$.

The n -th radial eigenfunction satisfies the equation

$$\frac{\partial^2 \Phi_\beta^n(\chi)}{\partial \chi^2} + 2 \coth \chi \frac{\partial \Phi_\beta^n(\chi)}{\partial \chi} + \left(k^2 - \frac{n(n+1)}{\sinh^2 \chi} \right) \Phi_\beta^n(\chi) = 0. \quad (\text{B3})$$

From the Φ_β^n we may construct the quantity

$$\Omega_\nu \equiv \frac{\Psi_\nu}{(\sinh \chi)^\nu} = \frac{\Phi_\beta^n(\chi)}{(\sinh \chi)^n}, \quad (\text{B4})$$

where $\nu = n + 1/2$ and Ψ_ν , defined by $\Psi_\nu \equiv \sqrt{\sinh \chi} \Phi_\beta^n(\chi)$, expresses a relationship similar to that between the spherical (half-integer) and ordinary (integer) Bessel functions. Then, Ω_ν satisfies the equation

$$\frac{\partial^2 \Omega_\nu}{\partial \chi^2} + (2\nu + 1) \coth \chi \frac{\partial \Omega_\nu}{\partial \chi} + [k^2 + (\nu - 1/2)(\nu + 3/2)] \Omega_\nu = 0. \quad (\text{B5})$$

Next, we use the addition rule for a hyper-sphere (B2) to write (B5) in terms of derivatives with respect to $\chi_{\mathbf{u}}$ and σ , assuming without loss of generality that $\chi_{\mathbf{v}}$ is constant. Hence, we obtain the equation

$$\begin{aligned} \frac{1}{\sinh^2 \chi_{\mathbf{u}}} \left[\frac{\partial^2}{\partial \sigma^2} + 2\nu \cot \sigma \frac{\partial}{\partial \sigma} \right] \Omega_\nu + \frac{\partial^2 \Omega_\nu}{\partial \chi_{\mathbf{u}}^2} + (2\nu + 1) \coth \chi_{\mathbf{u}} \frac{\partial \Omega_\nu}{\partial \chi_{\mathbf{u}}} \\ + [k^2 + (\nu - 1/2)(\nu + 3/2)] \Omega_\nu = 0. \end{aligned} \quad (\text{B6})$$

We now assume that Ω_ν can be expanded as $\Omega_\nu = \sum_{m=n}^{\infty} B_m C_m^\nu(\cos \sigma)$ where B_m is independent of σ , and the $C_m^\nu(\cos \sigma)$ is a (Gegenbauer) polynomial of degree m in $\cos \sigma$.

Formally, it may be shown [38] that $\left\{ \frac{\partial^2}{\partial \sigma^2} + 2\nu \cot \sigma \frac{\partial}{\partial \sigma} \right\} C_m^\nu(\cos \sigma)$ is a constant multiple of $C_m^\nu(\cos \sigma)$, so that $C_m^\nu(\cos \sigma)$ may be taken to be the coefficient of a^m in the series expansion of $(1 - 2a \cos \sigma + a^2)^{-\nu}$.

From the properties of these Gegenbauer functions $C_m^\nu(\cos \sigma)$, [36], we thus arrive at the following equation for B_m , as a function of $\chi_{\mathbf{u}}$:

$$\frac{\partial^2 B_m}{\partial \chi_{\mathbf{u}}^2} + (2\nu + 1) \coth \chi_{\mathbf{u}} \frac{\partial B_m}{\partial \chi_{\mathbf{u}}} + [k^2 + (\nu - 1/2)(\nu + 3/2)] B_m - \frac{m(2\nu + m)}{\sinh^2 \chi_{\mathbf{u}}} B_m = 0, \quad (\text{B7})$$

which has solutions $B_m \propto \frac{\Phi_\beta^{n+m}(\chi_{\mathbf{u}})}{(\sinh \chi_{\mathbf{u}})^n}$, and so, appealing to the symmetry between $\chi_{\mathbf{u}}$ and $\chi_{\mathbf{v}}$, we have (B1).

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